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A DOUBLE-DUAL CHARACTERIZATION OF SEPARABLE BANACH SPACES CONTAINING l^1

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ABSTRACT

It is proved that a separable Banach space B contains a subspace isomorphic to l^1 if (and only if) there exists an element in B^{**}, the double-dual of B, which is not a weak* limit of a sequence of elements in B. Consequently B contains an isomorph of l^1 if (and only if) the cardinality of B^{**} is greater than that of the continuum.

1. Introduction

Our main result is as follows:

MAIN THEOREM. *A separable Banach space B contains a subspace isomorphic to l^1 if (and only if) there exists an element G in B^{**}, the double dual of B, so that there is no sequence (b_n) in B with the property that $f(b_n) \to G(f)$ for all f in B^*, the dual of B.*

Suppose that B is separable. A simple cardinality argument shows that the hypotheses apply provided that the cardinality of B^{**} is larger than that of the continuum. On the other hand, since card $(l^1)^{**} = 2^c$, the Hahn-Banach theorem showes that if B contains an isomorph of l^1, then card $B^{**} = 2^c$. Also, no element of $(l^1)^{**} \backslash l^1$ is a weak* limit of a sequence of elements of l^1, so the "only if" assertion is immediate. It is proved in [12] that the hypotheses of the Main Theorem apply provided there exists a bounded sequence in B^{**} with no weak*-convergent subsequence. The latter generalizes the fundamental result of [11]: a Banach space contains an isomorph of l^1 if and only if contains a bounded sequence with no weak-Cauchy subsequence. These results may be summarized as follows:

Let B be a separable Banach space. Then the following five assertions are equivalent:

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(1) *B contains no isomorph of l^1.*

(2) *Every element of B^{**} is a weak* limit of a sequence in B.*

(3) Card B^{**} = card B.

(4) *Every bounded sequence in B has a weak-Cauchy subsequence.*

(5) *Every bounded sequence in B^{**} has a weak*-convergent subsequence.*

James has constructed a separable Banach space B with B^* non-separable, such that B contains no isomorph of l^1 [6] (see [3] for other examples). Thus James' example, together with the equivalence of (1) and (4), answers the following question of Banach in the negative (see the last question on page 243 of [2]): if B is separable and B^* is non-separable, does B contain a bounded sequence with no weak-Cauchy subsequence? Actually, prior to the discovery of the results of [11], Charles Stegall established that James' example satisfied properties (2)–(5). It seems quite surprising that all five properties are equivalent in the general case.

Of course James' example shows the falsehood of a natural conjecture. However, the above results show that if B is separable and B^* is sufficiently non-separable (so that any of (2)–(5) are violated), then B indeed contains l^1. We refer the reader to [12] for a summary of the currently known characterizations of Banach spaces containing l^1. The characterizations given above differ from the ones known prior to [11] in that the Banach space l^1 itself does not appear in the hypotheses. The techniques of [11] are combinatorial in nature. The arguments of the present paper and [12], on the other hand, are in the main topological: they turn heavily on the Baire category theorem and especially the following remarkable result published by Baire in 1889 [1]:

THE BAIRE CHARACTERIZATION THEOREM. *Let K be a non-empty compact metric space and f a real-valued function defined on K. Then f belongs to the first Baire class on K (i.e. f is a point-wise limit of a sequence of continuous functions on K) if and only if for every non-empty closed subset M of K, $f|M$ has a point of continuity relative to the topological space M.*

The "only if" assertion is a common exercise in most beginning graduate courses in analysis. However, it is the "if" assertion which we use; see pp. 288–289 of [5] for an elegant exposition.

REMARK. The Baire characterization theorem actually holds for a much wider class of topological spaces K. The following three hypotheses are adequate:

(1) K is normal.

(2) Every closed non-empty subset of K is of the second category in itself.

(3) K has no strictly descending transfinite sequence of closed subsets (i.e., there is no family $\{K_\alpha : \alpha < \omega_1\}$ of closed subsets of K, indexed by the first uncountable ordinal ω_1, with $K_\alpha \subsetneq K_\beta$ for all $\beta < \alpha < \omega_1$).
Thus, the theorem holds for any complete separable metric space K or any compact Hausdorff space K satisfying (3).

2. Proof of the main result

The case of complex Banach spaces follows from the case of real ones (see [11] and also [4]), so we restrict our attention to real spaces only. Our main result follows easily from the three lemmas which follow, a proposition in [11], and the Baire characterization theorem. We shall first show the deduction of our Main Theorem from these ingredients. Let B and G be as in the statement of the main theorem, and let K denote the unit-ball of B^* in its weak* topology. K is compact metrizable; Lemma 1 yields $G|K$ does not belong to the first Baire class on K. By the Baire characterization theorem, there is a closed non-empty subset M of K such that $G|M$ has no points of continuity. By Lemma 2, there is a closed non-empty subset L of M, and real numbers r, δ with $\delta > 0$, satisfying (*) of Lemma 2. Now suppose $\|G\| = 1$; by Goldstine's theorem, there is a net in the unit ball of B which converges weak* to B. Consequently $G|L$ is in the closure of the subset $\mathcal{G}$ of continuous functions on L defined by: $\mathcal{G} = \{g \in C(L)$: there is a b in B with $\|b\| \leqq 1$ and $g(x) = x(b)$ for all $x \in L\}$. By our Lemma 3 and Proposition 4 of [11], there is a sequence (g_j) in $\mathcal{G}$ independent enough to be equivalent to the usual basis of l^1 in the sup-norm on L (see the Remark following the statement of Lemma 3). Now simply choose (b_n) in the unit ball of B with $g_j(x) = x(b_j)$ for all j and x in L. Then the closed linear span of the b_n's is the desired subspace of B which is isomorphic to l^1.

We now proceed to the three lemmas and their proofs. We first need the

Definition. An element of the double dual X^{**} of a Banach space X is called a Baire-1 member of X^{**} provided it is the weak* (i.e. X^*) limit of a sequence of elements of X. The set of all Baire-1 elements of X^{**} shall be denoted $\mathcal{B}_1(X)$.

$\mathcal{B}_1(X)$ was introduced by McWilliams in [8];† he used the notation $K(X)$. Suppose K is a compact Hausdorff space. As observed by McWilliams, if

† See note added in proof.

$X = C(K)$, then the set of bounded first-Baire class functions on K may be identified with $\mathscr{B}_1(X)$ (where the Baire-1, or first Baire class functions on K, are those which are equal to a point-wise limit of a sequence of continuous functions on K). Indeed, regard K as being canonically imbedded in X^*; the map $f \to f|K$ assigns to each Baire-1 element f of X^{**}, a bounded Baire-1 function on K. If, conversely, g is a bounded Baire-1 function on K, then there is a sequence (g_n) of continuous functions converging point-wise to g. By suitably truncating, we can assume that $\overline{\lim}_n \|g_n\|_\infty = \|g\|_\infty$. Thus, if μ is a regular signed finite Borel measure on K, we have that $\int g_n d\mu \to \int g d\mu$, by the bounded convergence theorem. Identifying X^* with the space of all such μ by the Riesz representation theorem, we thus easily obtain that the above map is a surjective isometry.

Our first lemma follows easily from a theorem of Choquet ([13]; see also pp. 100–105 of [14]). We prefer, however, to give a self-contained argument, rather different from the one in [14], which in fact can be used to deduce Choquet's theorem.

LEMMA 1. *Let X be a Banach space and K denote the closed unit ball of X^* in its weak* topology. Then $f \in X^{**}$ is a Baire-1 member of X^{**} if (and only if) $f|K$ is a Baire-1 function on K.*

We first need a simple sublemma. For Banach spaces $A \subset B$, we use the notation $A^\perp = \{f \in B^*: f(a) = 0 \text{ for all } a \in A\}$.

SUBLEMMA. *Let X be a subspace of the Banach space B. Identify X^{**} with the subspace $X^{\perp\perp}$ of B^{**} and let $G \in X^{**}$ be a Baire-1 member of B^{**}. Then G is a Baire-1 member of X^{**}; in fact, if $\|G\| = 1$, then there is a sequence of elements of X of norm one which converges weak* to G.*

PROOF. Let $b_n \to G$ weak* with b_n in B for all n and assume $\|G\| = 1$. We need only show that $d(S_X, \overline{\text{co}}\{b_N, b_{N+1}, \cdots\}) = 0$ for all N(S_X denotes the closed unit ball of X; $\overline{\text{co}}S$ the closed convex hull of a set S, and $d(A, S)$ the distance between two subsets A and S of B). For then we can choose sequences (x_n) in S_X and $(\bar{b}_n)$ in B, the $\bar{b}_n$'s being "far-out" convex combinations of the b_n's, with $\|x_n - \bar{b}_n\| \to 0$. Since $\bar{b}_n \to G\ B^*$, we have that $x_n \to G\ B^*$ also; so $x_n \to G\ X^*$ by the Hahn-Banach theorem.

Now the geometrical form of the Hahn-Banach theorem asserts that if two convex sets in a locally convex space are a positive distance apart, then they may be strictly separated by a continuous linear functional (see p. 118 of [7]). (Convex sets E and F are said to be a "positive distance apart" if there is a neighborhood U of the origin with $(E + U) \cap F = \varnothing$; this obviously coincides with the metric notion in case the overlying space is Fréchet.) Consequently, if for some N, $d(S_X, \overline{\text{co}}\{b_N, b_{N+1}, \cdots\}) > 0$, there is an $f \in B^*$ so that

$$\operatorname{Sup}_{x \in S_X} f(x) < \inf_{j \geq N} f(b_j).$$

By Goldstine's theorem,

$$|G(f)| \leq \sup_{x \in S_X} |f(x)| < \inf_{j \geq N} f(b_j) \leq \lim_{j \to \infty} f(b_j) = G(f),$$

a contradiction. ∎

REMARK. Our proof of this Sublemma immediately gives the following result: *Let D be a convex subset of a Banach space X, $f \in \mathscr{B}_1(X)$, and suppose f is in the weak* closure of D. Then there exists a sequence in D which converges weak* to f.*

PROOF OF LEMMA 1. Let $f \in X^{**}$ be such that $f \mid K$ is a Baire-1 function on K. It suffices to prove that $f \in \mathscr{B}_1(X)$ under the assumption that $X = C(\Omega)$ for some compact Hausdorff space Ω. For, once this is done, we may choose an Ω (e.g. K itself) and an into-isometry $T: X \to C(\Omega)$. Letting E denote the unit ball of $C(\Omega)^*$, we have that $T^* \mid E$ is a continuous map of E onto K, by the Hahn-Banach theorem (where T^* denotes the adjoint of the map T). It follows that $T^{**}f$ is a Baire-1 function on E, for $T^{**}f = f \circ T^*$; so choosing $h_n \in C(K)$ with $h_n \to f$ pointwise, $h_n \circ T^* \to f \circ T^*$ pointwise on E. Thus we obtain that $T^{**}f \in \mathscr{B}_1(C(\Omega))$. But $T^{**}f \in (TX)^{\perp\perp}$; hence by the Sublemma, $T^{**}f \in \mathscr{B}_1(TX)$, whence $f \in \mathscr{B}_1(X)$.

We now assume that $X = C(\Omega)$ for some compact Hausdorff space Ω. Then the result of Choquet's mentioned above implies that $f \in \mathscr{B}_1(X)$. Alternatively, we argue as follows:

We identify X^* with $M(\Omega)$, the space of all regular signed Borel measures on Ω. For $\mu \in M(\Omega)$, let supp μ denote the set of all $x \in \Omega$, so that $|\mu|(\mathscr{U}) > 0$ for every open neighborhood $\mathscr{U}$ of x; of course supp μ is a closed subset of Ω. For S a closed subset of Ω, let $\mathscr{P}(S)$ denote the set of all probability measures $\mu \in M(\Omega)$ so that supp $\mu \subset S$; then $\mathscr{P}(S)$ is a weak* closed subset of K. Our strategy is to show that if $f \notin \mathscr{B}_1(X)$, then there exists a μ so that $f \mid \mathscr{P}(S)$ has no points of continuity in $\mathscr{P}(S)$, where $\mu \in \mathscr{P}(\Omega)$ and $S = \operatorname{supp} \mu$. This contradicts the assumption that f is a Baire-1 function on K, for the "only if" part of the Baire characterization theorem holds on arbitrary compact Hausdorff spaces. Toward this end, we observe that the following sets are weak* dense in $\mathscr{P}(S)$: $\mathscr{P}_d(S)$, the set of all purely atomic members of $\mathscr{P}(S)$; and $\mathscr{P}_\mu$, where $\mathscr{P}_\mu$ denotes the set of all $\lambda \in \mathscr{P}(\Omega)$, so that λ is absolutely continuous with respect to μ (notation: $\lambda \ll \mu$). Indeed, if $Y = \mathscr{P}_d(S)$ or $Y = \mathscr{P}_\mu$, then Y is convex and $\|f\|_\infty = \sup_{\nu \in Y} |\int f d\nu|$ for all $f \in C(S)$; hence Y is weak*-dense in $\mathscr{P}(S)$ by the Hahn-Banach theorem.

Let us regard Ω as being canonically imbedded in K. Then $f|\Omega$ is a Baire-1 function on Ω; it follows that defining $g \in X^{**}$ by $g(\mu) = \int_\Omega f(\omega) d\mu(\omega)$ for all $\mu \in M(\Omega)$, then $g \in \mathscr{B}_1(X)$ and hence $h = f - g$ is a Baire-1 function when restricted to K. We need only show that $h = 0$; our definition of h shows that

$$h(\mu) = 0 \text{ for all } \mu \in \mathscr{P}_d(\Omega). \tag{1}$$

Now suppose that $h \neq 0$. Since every $\mu \in M(\Omega)$ is a difference of multiples of elements of $\mathscr{P}(\Omega)$, there is a $\nu \in \mathscr{P}(\Omega)$ with $h(\nu) \neq 0$; by multiplying h by -1 if necessary, we may assume that $h(\nu) > 0$. Let Z denote the space of all $\lambda \in M(\Omega)$ with $\lambda \ll \nu$; we identify Z with $L^1(\nu)$ by the Radon-Nikodym theorem. Now $h|Z$ is a bounded linear functional on Z; hence, by the Riesz representation theorem, there is a bounded Borel-measurable function ϕ so that

$$h(\lambda) = \int \phi d\lambda \quad \text{for all } \lambda \in Z. \tag{2}$$

In particular, $h(\nu) = \int \phi d\nu > 0$, hence $\int \phi^+ d\nu > 0$, where ϕ^+ denotes the positive part of ϕ. Now choose a positive number c so that $\nu(E) > 0$, where $E = \{\omega \in \Omega : \phi(\omega) \geqq c\}$. It follows that if $\lambda \in \mathscr{P}(\Omega)$ is such that $\lambda(\sim E) = 0$, then

$$\int \phi d\lambda = \int_E \phi d\lambda \geqq c. \tag{3}$$

Now let $\mu \in \mathscr{P}(\Omega)$ be defined by

$$\mu(B) = \frac{\nu(B \cap E)}{\nu(E)}$$

for all Borel sets B. It follows from (2) and (3) that

$$h(\lambda) \geqq c \quad \text{for all} \quad \lambda \in \mathscr{P}_\mu. \tag{4}$$

We have now reached our goal. Let $S = \operatorname{supp} \mu$. Then $h \geqq c$ on a dense subset of $\mathscr{P}(S)$, namely $\mathscr{P}_\mu(S)$, by (4); $h = 0$ on another dense subset of $\mathscr{P}(S)$, namely $\mathscr{P}_d(S)$, by (1); hence $h|\mathscr{P}(S)$ has no points of continuity in $\mathscr{P}(S)$, contradicting the fact that h is a Baire-1 function on K. ∎

In the following remarks, let X and K be as in Lemma 1.

1. Let $X = C([0,1])$ and $f \in X^{**}$. Lemma 1 shows that if f is a Baire-1 function on K, then f may be identified with a Baire-1 function on $[0,1]$; precisely,

$$f(\mu) = \int_{[0,1]} f d\mu \quad \text{for all} \quad \mu \in M([0,1]).$$

This is an immediate consequence of Choquet's theorem mentioned above. An example of Choquet (see pp. 104–105 of [14]) yields the following: There exists an $f \in X^{**}$ so that f is in the 2nd-Baire class on K, yet f cannot be identified with a Borel-measurable function on $[0,1]$. We define f by $f(\mu) = \mu_s([0,1])$, where for any measure $\mu \in M([0,1],\mu_s$ denotes the singular part of μ with respect to Lebesgue measure m. (Part of our proof of Lemma 1 boils down to the fact that f is not a Baire-1 function on K.) It's obvious that f cannot be identified with a Borel-measurable function on $[0,1]$ since $f(m) = 0$ yet $f(\mu) = 1$ for all $\mu \in P_d([0,1])$. Now for each n, define the function f_n on K by

$$f_n(\mu) = \sup\left\{\int \phi d\mu : \phi \in C([0,1]),\ 0 \leqq \phi \leqq 1 \quad \text{and} \quad \int_0^1 \phi(t) dm(t) < \frac{1}{n}\right\}.$$

f_n is Baire-1 on K; indeed, f_n is lower-semi-continuous, being the supremum of a family of continuous functions on K. Then $f(\mu) = \lim_{n\to\infty} f_n(\mu) - f_n(-\mu)$; hence f is a point-wise limit of a sequence of Baire-1 functions, so f is Baire-2 on K.

2. Since $\mathscr{B}_1(K)$, the space of all bounded real-valued functions on K of the first Baire-class, is a complete Banach space; and $\mathscr{B}_1(X)$ may be identified with $\mathscr{B}_1(K) \cap X^{\perp\perp}$, we obtain McWilliams' result [8] that $\mathscr{B}_1(X)$ is norm-closed in X^{**}. The essential ingredient in the proof that $\mathscr{B}_1(X)$ is complete, is already contained in our proof of the sublemma; the Hahn-Banach argument replaces the more classical truncation argument. On the other hand, McWilliams showed in [8] that the "in fact" assertion of the sublemma follows from the assumption that $G \in \mathscr{B}_1(X)$.

3. One may introduce higher Baire-classes of members of X^{**} as follows: for α a limit ordinal, let $\mathscr{B}_\alpha(X) = \cup_{\tau<\alpha} \mathscr{B}_\tau(X)$; for general α, let $\mathscr{B}_{\alpha+1}(X)$ equal the set of elements of X^{**} which are equal to a weak* limit of a sequence of elements of $\mathscr{B}_\alpha(X)$. Put $\mathscr{B}_0(X) = X$ where X is regarded as being contained in X^{**}. These were introduced by McWilliams in [10];† as he observed there, in the case where $X = C(S)$ for some compact Hausdorff space S, $\mathscr{B}_\alpha(X)$ may be identified with $\mathscr{B}_\alpha(S)$, the Banach space of all bounded functions on S of the αth Baire-class. However, the analogue of the sublemma fails in the context of these higher Baire-classes. Indeed, let X and f be as in the example in Remark

† See note added in proof.

1. Let $T: X \to C(K)$ be the natural isometry. Then $f \notin \mathscr{B}_\alpha(X)$ for any $\alpha < \omega_1$, yet $T^{**}f \in \mathscr{B}_2(C(K))$. Hence $\mathscr{B}_\alpha(TX) \neq (TX)^{\perp\perp} \cap \mathscr{B}_\alpha(C(K))$ for any ordinal $\alpha \geqq 2$. Of course, $\mathscr{B}_2(X)$ is norm-closed. McWilliams has exhibited in [9] a space X so that $\mathscr{B}_2(X)$ is not norm-closed. Hence also for this $X, \mathscr{B}_2(TX) \neq (TX)^{\perp\perp} \cap \mathscr{B}_2(C(K))$.

Our next result applies the Baire category theorem to show that if a function on a compact Hausdorff space has no points of continuity, then there is a subset of the space where the function is badly discontinuous; in particular, its relative oscillation at all points is absolutely bounded away from zero.

LEMMA 2. *Let K be a non-empty compact Hausdorff space and f a bounded real-valued function on K having no points of continuity. Then there is a closed non-empty subset L of K and real numbers r, δ with $\delta > 0$ so that:*

(∗) *For every non-empty relatively open subset U of L, there are y and z in U with $f(y) > r + \delta$ and $f(z) < r$.*

PROOF. For each positive integer n, let A_n equal the set of all x in K so that, if U is a neighborhood of x, there are y, z in U with $f(y) - f(z) > 1/n$. Since f has no points of continuity, $K = \bigcup_{n=1}^\infty A_n$. By definition the A_n's are all closed, so by the Baire category theorem some A_{n_0} has non-empty interior, U_0. Let $K_0 = \bar{U}_0$ and $\delta = 1/n_0$. We now have that, for any non-empty relatively open subset U of K_0, $U \cap U_0$ is a non-empty open subset of K_0, and hence there exist y and z in U with $f(y) - f(z) > \delta$. Now let $(r_n)_{n=1}^\infty$ be an enumeration of the rationals and for each n set B_n equal to the set of all x in K_0 so that, if U is a neighborhood of x, there are y, z in $U \cap K_0$ with $f(z) < r_n$ and $r_n + \delta < f(y)$. Since (∗) holds for every non-empty open subset of K_0, $K_0 = \bigcup_{n=1}^\infty B_n$. Again, all the B_n's are closed by definition, so a second application of the category theorem implies B_{n_1} has non-empty interior V for some n_1. Then, putting $L = \bar{V}$ and $r = r_{n_1}$, we have that (∗) holds. (We have also incidentally obtained that L may be chosen equal to the closure of an open subset of K.) ∎

Our next result gives the final step in the production of our l^1-sequence. We recall that a sequence $(A_n, B_n)_{n=1}^\infty$ of pairs of subsets of some set, is called independent provided $A_n \cap B_n = \varnothing$ for all n, and for any two disjoint finite subsets F_1 and F_2 of the positive integers,

$$\bigcap_{n \in F_1} A_n \cap \bigcap_{n \in F_2} B_n \neq \varnothing.$$

LEMMA 3. *Let f, L, r, and δ satisfy the conclusion of Lemma 2. Let $\mathscr{G}$ be a bounded subset of the continuous functions on L and assume that f is in the* *closure of $\mathscr{G}$ in the topology of pointwise convergence. Then there exists a sequence (g_n) of elements of $\mathscr{G}$ so that putting $A_n = \{x \in L : g_n(x) > r + \delta\}$ and $B_n = \{x \in L : g_n(x) < r\}$ for all n, then (A_n, B_n) is independent.*

REMARK. The hypotheses imply that L is a perfect set such that for any $\varepsilon > 0$ and $\lambda_1, \cdots, \lambda_k$ in L, there is a g in $\mathscr{G}$ with $|g(\lambda_i) - f(\lambda_i)| < \varepsilon$ for all $1 \leqq i \leqq k$. If (g_n) satisfies the conclusion, then by Proposition 4 of [11], the closed linear span of the g_n's, in the sup norm, is isomorphic to l^1; in fact, for any k and real numbers $c_1, \cdots, c_k$,

$$\sup_{x \in L} \left| \sum c_i g_i(x) \right| \geqq \frac{\delta}{2} \sum |c_i|.$$

PROOF OF LEMMA 3. Choose y_1, y_2 in L so that $f(y_1) > r + \delta$ and $f(y_2) < r$. Then choose g_1 in $\mathscr{G}$ so that $g_1(y_1) > r + \delta$ and $g_1(y_2) < r$. Let $n > 1$ and assume $g_1, \cdots, g_{n-1}$ have been chosen so that $\bigcap_{i=1}^{n-1} \varepsilon_i A_i \neq \varnothing$ for all choices of signs $\varepsilon_i = \pm 1$, where $\eta A_i = A_i$ if $\eta = +1$, $\eta A_i = B_i$ if $\eta = -1$, and the A_i, B_i's are as defined in the statement of the lemma. For each such choice of signs $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_{n-1})$, $\bigcap_{i=1}^{n-1} \varepsilon_i A_i$ is a non-empty open set in L and we may pick $y_1^\varepsilon, y_2^\varepsilon$ in $\bigcap_{i=1}^{n-1} \varepsilon_i A_i$ so that $f(y_1^\varepsilon) > r + \delta$ and $f(y_2^\varepsilon) < r$. Again we may choose g_n in $\mathscr{G}$ so that $g_n(y_1^\varepsilon) > r + \delta$ and $g_n(y_2^\varepsilon) < r$ for all 2^{n-1} choices of ε. It follows that $\bigcap_{i=1}^{n} \varepsilon_i A_i \neq \varnothing$ for all choices of ε_i; the sequence $(g_j)_{j=1}^{\infty}$ thus constructed satisfies the conclusion of the lemma. ■

This completes the proof of our main result. It is known that its conclusion fails for non-separable spaces in general. Indeed, if Γ is an uncountable set and $B = c_0(\Gamma)$, then B contains no isomorph of l^1, yet $\mathscr{B}_1(B)$ may be identified with the proper subset of $l^\infty(\Gamma) = (c_0(\Gamma))^{**}$ consisting of all functions which vanish off a countable subset of Γ.

REMARK. It follows from our Main Theorem and the work of Choquet that, if X is separable and does not contain an isomorph of l^1, then every bounded ω^* closed convex subset A of X^* is the norm closed convex hull of the set of its extreme points, E. Indeed, if $x^* \in A \setminus \overline{\text{co}}(E)$, then there is an $f \in X^{**}$ so that $f(x^*) > \sup_{e \in E} f(e)$. But this is impossible, for there is a probability measure μ on A so that $\mu(A \setminus E) = 0$ and

$$f(x^*) = \int_A f \, d\mu = \int_E f \, d\mu$$

(see Sections 3 and 12 of [14]).

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